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Norm of the Bernstein Left Quasi-interpolant Operator

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A Bernstein quasi-interpolant operator $B_n^{(k)}$ has been introduced by Sablonniere (*in* "Multivariate Approximation Theory, Vol. IV" (C. K. Chui, W. Schempp, and K. Zeller, Eds.), Birkhauser, Basel, 1989). In this paper we show that for fixed k the norm $\|B_n^{(k)}\|_\infty$ is uniformly bounded in n. This answers a conjecture of Sablonniere.

1. INTRODUCTION

Let f be a function defined on $[0, 1]$. The Bernstein operator B_n is defined by

$$B_n f = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{ni}(x),$$

where $p_{ni} = \binom{n}{i} x^i(1-x)^{n-i}, 0 \leqslant i \leqslant n$. It is well known the approximation order of B_n is $O(1/n)$. To obtain faster convergence various Bernstein-type quasi-interpolants have been introduced. Recently P. Sablonniere introduced such an operator [4]. To illustrate the operator we recall some notations (see [4]).

Let $\mathbb{P}_n$ denote the space of polynomials of degree at most n. On the space $\mathbb{P}_n$ the operator B_n can be considered as a linear differential operator [3, 4]

$$B_n = \sum_{i=0}^{n} \beta_i^n D^i, \tag{1}$$

where $\beta_i^n \in \mathbb{P}_i$ are defined by the recurrence relation

$$n(i+1)\, \beta_{i+1}^n(x) = X(D\beta_i^n(x) + \beta_{i-1}^n(x)),$$

$$\beta_0^n = 1, \qquad \beta_1^n = 0,$$

where (and in the following) $X = x(1-x)$.

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Because B_n is a one-to-one mapping on $\mathbb{P}_n$ there exists the inverse operator B_n^{-1} defined on $\mathbb{P}_n$ which can also be considered as a linear differential operator

$$B_n^{-1} = \sum_{j=0}^{n} \alpha_j^n D^j,$$

where the coefficients α_j^n are defined by the recurrence relation

$$\begin{gathered} \sum_{r=0}^{s} \delta_{sr}^n \alpha_r^n = 0, \\ \alpha_0^n = 1, \qquad \alpha_1^n = 0, \end{gathered} \tag{2}$$

where $\delta_{sr}^n = \sum_{i=0}^{r} \binom{r}{i} D^i \beta_{i+s-r}^n$.

For $0 \leqslant k \leqslant n$, [4] introduced the truncated inverse of B_n

$$A_n^{(k)} = \sum_{j=0}^{k} \alpha_j^n D^j$$

and defined the so-called left Bernstein quasi-interpolant $B_n^{(k)}$ (of order k)

$$B_n^{(k)} = A_n^{(k)} B_n.$$

That is,

$$\begin{aligned} B_n^{(k)} &= A_n^{(k)} B_n \\ &= \sum_{j=0}^{k} \alpha_j^n D^j B_n. \end{aligned}$$

In order to investigate the convergence of the left Bernstein quasi-interpolant in $C[0, 1]$, we want to know whether the norm of the operator $B_n^{(k)}$ is bounded for k fixed. Here we consider $C[0, 1]$ as a Banach space $(C[0, 1], \|\cdot\|_\infty)$ with the norm $\|f\|_\infty = \sup_{0 \leqslant x \leqslant 1} |f(x)|$ for $f(x) \in C[0, 1]$ and $B_n^{(k)}$ as a linear operator $B_n^{(k)}$: $(C[0, 1], \|\cdot\|_\infty) \to (C[0, 1], \|\cdot\|_\infty)$. For $k = 0$, $k = 1$, we have

$$B_n^{(0)} = B_n^{(1)} = B_n.$$

Hence

$$\|B_n^{(k)}\|_\infty \leqslant 1, \qquad k = 0, 1.$$

For $k = 2$,

$$\|B_n^{(2)}\|_\infty \leqslant 3$$

is proved in [4]. As to what happens for $k \geqslant 3$, P. Sablonniere has the following conjecture [4]: for fixed k, the norm $\|B_n^{(k)}\|_\infty$ is uniformly (about n) bounded. Then based on the conjecture he has proved a convergence result of $B_n^{(k)}$. And it is said that the experimental results support the conjecture. Thus it is crucial whether the conjecture is true or not. In this short note we will prove this conjecture.

2. Preliminaries

From the expression of B_n, it is easy to derive

$$\beta_i^n(x) = \frac{1}{i!} B_n((\cdot - x)^i)(x). \tag{3}$$

The properties of $B_n((\cdot - x)^i)(x)$ have been extensively investigated in papers involving Bernstein polynomials (for example, [1, 2, 5]). We recall the recurrence relation of $B_n((\cdot - x)^i)(x) \equiv T_{n,i}(x)$:

$$T_{n,0}(x) = 1,$$

$$T_{n,1}(x) = 0,$$

$$T_{n,i+1}(x) = \frac{X}{n}(T'_{n,i}(x) + iT_{n,i-1}(x)).$$

By induction it is easy to check that

$$T_{n,2m}(x) = C_0^{(m)}(x)\frac{X^m}{n^m} + C_1^{(m)}(x)\frac{X^{m-1}}{n^{m+1}} + \cdots + C_{m-1}^{(m)}(x)\frac{X}{n^{2m-1}},$$

$$T_{n,2m+1}(x) = D_0^{(m)}(x)\frac{X^m}{n^{m+1}} + D_1^{(m)}(x)\frac{X^{m-1}}{n^{m+2}} + \cdots + D_{m-1}^{(m)}(x)\frac{X}{n^{2m}}.$$

Set $s' = [(s+1)/2]$. We can write

$$T_{n,s}(x) = F_{s,s-s'-1}(x)\frac{X}{n^{s-1}} + F_{s,s-s'-2}(x)\frac{X^2}{n^{s-2}} + \cdots + F_{s,0}(x)\frac{X^{s-s'}}{n^{s'}}, \tag{4}$$

where $F_{s,j}(x)$ are polynomials independent of n (just as are $C_j^{(s)}(x)$ and $D_j^{(s)}(x)$). Repeated differentiation of (4) shows that

$$D^j T_{n,s}(x) = F_{s,s-s'-1}^j(x)\frac{X^{1-j}}{n^{s-1}} + F_{s,s-s'-2}^j(x)\frac{X^{2-j}}{n^{s-2}} + \cdots + F_{s,0}^j(x)\frac{X^{s-s'-j}}{n^{s'}}, \tag{5}$$

where we make the convention $X^{\alpha}=1$ for integer $\alpha<0$. $F^{j}_{s,i}(x)$ are polynomials bounded uniformly in n.

LEMMA 1. *δ^n_{sr} is defined in (2). Then for $r\leqslant s$ we have*

$$\delta^n_{sr}(x)=G^{sr}_{s-1}(x)\frac{1}{n^{s-1}}+G^{sr}_{s-2}(x)\frac{1}{n^{s-2}}+\cdots+G^{sr}_{s-r}(x)\frac{1}{n^{s-r}}$$
$$+G^{sr}_{s-r-1}(x)\frac{X}{n^{s-r-1}}+\cdots+G^{sr}_{[(s-r+1)/2]}(x)\frac{X^{s-r-[(s-r+1)/2]}}{n^{[(s-r+1)/2]}}. \qquad (6)$$

Note that

$$\delta^n_{ss-1}(x)=G^{ss-1}_{s-1}(x)\frac{1}{n^{s-1}}+G^{ss-1}_{s-2}(x)\frac{1}{n^{s-2}}+\cdots+G^{ss-1}_{1}(x)\frac{1}{n},$$

$$\delta^n_{s0}(x)=\frac{1}{s!}T_{n,s}(x),$$

Here $G^{sr}_j(x)$ are polynomials independent of n. In particular

$$\delta^n_{ss}=s!\binom{n}{s}n^{-s}. \qquad (6')$$

Proof. By the definition

$$\delta^n_{sr}=\sum_{i=0}^{r}\binom{r}{i}D^i\beta^n_{i+s-r}$$

and using (3) and (5), the rearrangement of the terms X yields (6). For (6′) see [4]. ∎

LEMMA 2. *We have $\alpha^n_0=1$, $\alpha^n_1=0$, and*

$$\alpha^n_s=H^{ns}_{s-1}(x)\frac{X}{n^{s-1}}+H^{ns}_{s-2}(x)\frac{X^2}{n^{s-2}}+\cdots+H^{ns}_{s'}(x)\frac{X^{s-s'}}{n^{s'}}, \qquad s\geqslant 2, \qquad (7)$$

where $s'=[(s+1)/2]$ and $H^{ns}_j(x)$ are functions uniformly bounded in n and $x\in[0,1]$.

Proof. By induction. For $s=2$ it is obvious by calculation. Now suppose (7) is true for $s\leqslant k-1$. By Lemma 1 we have

$$\delta^n_{kr}(x)=G^{kr}_{k-1}(x)\frac{1}{n^{k-1}}+G^{kr}_{k-2}(x)\frac{1}{n^{k-2}}+\cdots+G^{kr}_{k-r}(x)\frac{1}{n^{k-r}}$$
$$+G^{kr}_{k-r-1}(x)\frac{X}{n^{k-r-1}}+\cdots+G^{kr}_{[(k-r+1)/2]}(x)\frac{X^{k-r-[(k-r+1)/2]}}{n^{[(k-r+1)/2]}}$$

$$= \bar{G}^{kr}_{k-r}(x)\frac{1}{n^{k-r}} + G^{kr}_{k-r-1}(x)\frac{X}{n^{k-r-1}} + \cdots + G^{kr}_{[(k-r+1)/2]}(x)\frac{X^{k-r-[(k-r-1)/2]}}{n^{[(k-r+1)/2]}}.$$

Thus for $r \leqslant k-1$,

$$\delta^n_{kr}\alpha^n_r = \left(\bar{G}^{kr}_{k-r}(x)\frac{1}{n^{k-r}} + G^{kr}_{k-r-1}(x)\frac{X}{n^{k-r-1}} + \cdots + G^{kr}_{[(k-r+1)/2]}(x)\frac{X^{k-r-[(k-r+1)/2]}}{n^{[(k-r+1)/2]}}\right) \times \left(H^{nr}_{r-1}(x)\frac{X}{n^{r-1}} + H^{nr}_{r-2}(x)\frac{X^2}{n^{r-2}} + \cdots + H^{nr}_{r'}(x)\frac{X^{r-r'}}{n^{r'}}\right)$$

Note that

$$\left[\frac{k-r+1}{2}\right] + \left[\frac{r+1}{2}\right] \geqslant \left[\frac{k+1}{2}\right] = k'.$$

Hence with the rearrangement we can write $\delta^n_{kr}\alpha^n_r$ as

$$\delta^n_{kr}\alpha^n_r = L^{kr}_{k-1}(x)\frac{X}{n^{k-1}} + L^{kr}_{k-2}(x)\frac{X^2}{n^{k-2}} + \cdots + L^{kr}_{k'}(x)\frac{X^{k-k'}}{n^{k'}},$$

where obviously all $L^{kr}_j(x)$ are uniformly bounded in n and $x \in [0, 1]$.

Noting that

$$\delta^n_{kk} = k!\binom{n}{k} n^{-k},$$

we have

$$(\delta^n_{kk})^{-1} = \frac{n^k}{k!\binom{n}{k}} = \frac{n^k}{n(n-1)\cdots(n-k+1)} = \left(1+\frac{1}{n-1}\right)\left(1+\frac{2}{n-2}\right)\cdots\left(1+\frac{k-1}{n-k+1}\right) = 1 + K_n,$$

where K_n is bounded in n. Using the recurrence relation about α^n_k, we have

$$\delta^n_{kk}\alpha^n_k = -\sum_{r=0}^{k-1}\delta^n_{kr}\alpha^n_r.$$

It follows that (7) is true for $s = k$. ∎

3. Main Result

Now we are in a position to prove the main result about the norm of the left Bernstein quasi-interpolant.

Theorem. *Let*

$$B_n^{(k)} = A_n^{(k)} B_n$$
$$= \sum_{j=0}^{k} \alpha_j^n D^j B_n.$$

For fixed k, the norm $\|B_n^{(k)}\|_\infty$ is bounded. Namely, there is a constant M independent of n such that

$$\|B_n^{(k)}\|_\infty \leqslant M.$$

Proof.

$$B_n f = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{ni}.$$

It is well known that

$$D^j B_n f(x) = \frac{n!}{(n-j)!} \sum_{i=0}^{n-j} \Delta_{1/n}^j f\left(\frac{i}{n}\right) p_{n-j,i}(x)$$
$$\Delta_{1/n} = f\left(x+\frac{1}{n}\right) - f(x). \tag{8}$$

It has been proved that

$$X^r |D^{2r} Bf(x)| \leqslant C(r)\, n^r \,\|f\|_\infty, \qquad x \in [0, 1], \tag{9}$$

where C is a constant independent of n [2]. For integer j we write it as

$$j = s + 2r,$$

where s and r are integers, $0 \leqslant r \leqslant [j/2]$. Thus

$$D^j B_n f(x) = D^{2r}(D^s B_n f(x)) = D^{2r}\left(\frac{n!}{(n-s)!} \sum_{i=0}^{n-s} \Delta_{1/n}^s f\left(\frac{i}{n}\right) p_{n-s,i}(x)\right).$$

From (9) it follows that

$$X^r |D^j B_n f(x)| \leqslant C(j) \frac{n!}{(n-s)!} (n-s)^r \;\|\Delta_{1/n}^s f(\cdot)\|_\infty$$
$$\leqslant C_1(j)\, n^{s+r} \,\|f\|_\infty, \qquad x \in [0, 1]. \tag{10}$$

Now using Lemma 2, we obtain

$$|\alpha_j^n D^j B_n f(x)| \leqslant K\left(\frac{X}{n^{j-1}} + \cdots + \frac{X^{j-[(j+1)/2]}}{n^{[(j+1)/2]}}\right) |D^j B_n f(x)|,$$

where the constant $K \geqslant \max_{j' \leqslant s \leqslant j-1} \|H_s^{nj}(\cdot)\|_\infty$, independent of n. Noting that

$$j - \left[\frac{j+1}{2}\right] \leqslant \left[\frac{j}{2}\right],$$

and $j = s + 2r$, from (10) we have

$$\begin{aligned}\frac{X^r}{n^{j-r}} |D^j B_n f(x)| &\leqslant \frac{1}{n^{j-r}} C_1(j)\, n^{s+r} \|f\|_\infty \\ &\leqslant C_1(j) \|f\|_\infty, \qquad x \in [0, 1].\end{aligned}$$

Thus

$$|\alpha_j^n D^j B_n f(x)| \leqslant C_2(j) \|f\|_\infty, \qquad x \in [0, 1].$$

For fixed k we have

$$\begin{aligned}|B_n^{(k)} f(x)| &\leqslant \sum_{j=0}^{k} |\alpha_j^n D^j B_n f(x)| \\ &\leqslant \left(\sum_{j=0}^{k} C_2(j)\right) \|f\|_\infty, \qquad x \in [0, 1].\end{aligned}$$

Let $M = (\sum_{j=0}^{k} C_2(j)$. Then we obtain the desired result:

$$\|B_n^{(k)}\|_\infty \leqslant M.$$

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